

# Phase Transition in Conditional Curie-Weiss Model

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## Abstract

This paper proposes a conditional Curie-Weiss model as a model for opinion formation in a society polarized along two opinions, say opinions 1 and 2. The model comes with interaction strength  $\beta > 0$  and bias  $h$ . Here the population in question is divided into three main groups, namely:

1. Group one consisting of individuals who have decided on opinion 1. Let the proportion of this group be given by  $s$ .
2. Group two consisting of individuals who have chosen opinion 2. Let  $r$  be their proportion.
3. Group three consisting of individuals who are yet to decide and they will decide based on their environmental conditions. Let  $1 - s - r$  be the proportion of this group.

We show that the specific magnetization of the associated conditional Curie-Weiss model has a first order phase transition (discontinuous jump in specific magnetization) at  $\beta^* = (1 - s - r)^{-1}$ . It is also shown that not all the discontinuous jumps in magnetization will result in phase change. We point out how an extension of this model could serve as a random field Curie-Weiss model where the random field distribution has nonvanishing mean.

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# 1 Introduction and main results

Phase transition in a physical system corresponds to a change from one phase (behaviour) of the physical system to another through changes in the environment of the system. For instance the liquid-vapour transition of water when it is boiled. In the words of J. Willard Gibbs a phase transition is a singularity in thermodynamic behavior, i.e. singularity in the free energy of the system [4]. The nature of the singularity, according to P. Ehrenfest, determines the order of the phase transition. For instance, first order phase transition is associated with a discontinuous jump in some thermodynamic quantity called the order parameter of the system. An important theory that has offered a lot of insight into the study of phase transitions is the mean field theory. This theory says that each component of the physical systems feels the average influence of all the other components of the system. Johannes Diderik van der Waals was the first to derive mean field theory in the 1870s in his attempt at understanding the 1869 experimental data of T. Andrews [9] that showed a phase transition curve separating the liquid-vapour phases of fluids. In 1895, Pierre Curie observed that a ferromagnet admits a behaviour similar to that of fluids [11]. A mean field theory for ferromagnet was derived in 1907 by Pierre Weiss [12]. In the late 1960s Mark Kac developed a model for ferromagnets where every magnetic moment interacts with every other magnetic moment and this model is known in the literature as the Curie-Weiss (CW) model [2].

The Curie-Weiss model after its introduction has found other applications apart from what it was originally designed to do. Notable among these applications are formation of opinion in societies [13, 14], immigrants' integration [5, 6, 7], democratization [8], etc.

The present paper studies phase transition in Curie-Weiss model conditional on having certain minimum proportions of magnetic moments following each of the two possible spin alignments. This could serve as a model for opinion formation in a society segregated along two opinions. Here the minimum proportions are proportions of individuals who have fixed their orientation on one of the two opinions and they will never change their views. Phase transition here will imply emergence of consensus and lack of it will imply wild fluctuation of group opinion and never settling on a specific collective opinion or decision, i.e. non-consensus emergence [13].

In a followup paper, we will study the case where the minimum proportions are fixed according to some distribution and we ask for the effect of the disorder in these proportions on the phase transitions in the associated quenched and annealed models. This will provide a natural example of random field Ising type of model [15] with non centered random field. The present paper is the first step towards investigating these class of random field Ising models. Further, such models will naturally set the stage for studying spin models on site percolation clusters generated from a random process of assigning three different colours to the vertices of the underlying graph. The spins on one of the three clusters, generated from the three colours, will be fixed to +1, one of the remaining two will also be set to -1 and the spins on the remaining cluster could pick any of the spin values. We then ask the question of phase transition in Ising spin model on such a decorated graph.

The rest of the paper is organized as follows: In Section 1.1 we recall the Curie-Weiss model and collect some fact about it. Section 1.2 is devoted to defining our conditional

Curie-Weiss model. The main results of the paper are collected in Section 1.3. The results in Section 1.3 are discussed in Section 2 and the proofs of the main results are found in Section 3.

## 1.1 The Curie-Weiss Model

Let  $N$  be a positive integer,  $V_N = \{1, 2, \dots, N\}$  and  $E_N$  be the vertex set and the edge set respectively for a complete graph with  $N$  vertices. Denote by  $\Omega_N = \{-1, +1\}^N$ , the set of configurations of the system indexed by the elements of  $V_N$ . The Curie-Weiss model is a probability measure  $\mu_N$  on  $\Omega_N$  given by

$$\mu_N(\sigma) = \frac{1}{Z_N} \exp(-\beta H_N(\sigma)), \quad (1.1)$$

where for any  $\sigma \in \Omega_N$

$$\begin{aligned} H_N(\sigma) &= -\frac{1}{2N} \sum_{(i,j) \in E_N} \sigma_i \sigma_j - h \sum_{i=1}^N \sigma_i \\ &= -N \left[ \frac{1}{2} \left( \frac{1}{N} \sum_{i=1}^N \sigma_i \right)^2 - \frac{h}{N} \sum_{i=1}^N \sigma_i \right] - \frac{1}{2} \\ &= -N \left[ \frac{1}{2} m_N^2 + h m_N \right] - \frac{1}{2}. \end{aligned} \quad (1.2)$$

Here  $h$  is a real number and  $\beta$  is a positive real number. The quantity  $Z_N$  is a normalization term called the partition function of the model.  $h$  and  $\beta$  are the parameters of the model called the interaction bias and interaction strength respectively.  $h$  is usually called an external field and  $\beta$  is the inverse temperature. The function  $H_N$  on configurations set  $\Omega_N$  is called the Hamiltonian/ energy function of the model. The first term in the Hamiltonian turns to align pairs of spins, while the second turns to align spins in the direction of the external field  $h$ . It is known in the literature that the Curie-Weiss models undergoes a phase transition, namely the specific magnetization

$$m(\beta, h) = \lim_{N \rightarrow \infty} \frac{1}{N} \int_{\Omega_N} \left( \sum_{i=1}^N \sigma_i \right) d\mu_N(\sigma) \quad (1.3)$$

exhibits a discontinuity as  $h \rightarrow 0$  for  $\beta > 1$ . In fact, it is proved in Theorem IV.4.1 (b) of [1] that:

**Theorem 1.1** *For  $\beta > 0$ ,  $h \neq 0$  and  $0 < \beta \leq 1$ ,  $h = 0$ , the specific magnetization  $m(\beta, h)$  is equal to  $z(\beta, h)$ , where  $z(\beta, h)$  is the minimizer of the function*

$$i_{\beta, h}(z) = -\frac{1}{2}\beta z^2 - \beta h z + \frac{1-z}{2} \log(1-z) + \frac{1+z}{2} \log(1+z). \quad (1.4)$$

*In particular,*

$$m(\beta, \pm) = \lim_{h \rightarrow 0^\pm} m(\beta, h) = \begin{cases} z(\beta, 0) = 0 & \text{for } 0 < \beta \leq 1, \\ z(\beta, \pm) \neq 0 & \text{for } \beta > 1. \end{cases} \quad (1.5)$$

Thus when  $\beta > 1$ , any small change in  $h$ , no matter how small it is, that we make around  $h = 0$  will lead to a dramatic change in the behaviour of the system.

## 1.2 Conditional Curie-Weiss Model

For any positive integer  $N$ , let  $s_N$ ,  $r_N$  and  $t_N$  be positive real numbers such that  $r_N + s_N + t_N = 1$ ,  $r_N \rightarrow r$ ,  $s_N \rightarrow s$  and  $t_N \rightarrow t$  as  $N \rightarrow \infty$ . Further, let  $s_N$ ,  $r_N$  and  $t_N$  be such that there is a partition  $V_{N,r_N}$ ,  $V_{N,s_N}$  and  $V_{N,t_N}$  of  $V_N$  such that  $r_N = \frac{|V_{N,r_N}|}{N}$ ,  $s_N = \frac{|V_{N,s_N}|}{N}$  and  $t_N = \frac{|V_{N,t_N}|}{N}$ . Here we write  $|A|$  for the cardinality of a set  $A$ . Define a subset  $\Omega_{N,s,r}$  of  $\Omega_N$  consisting of configurations  $\sigma = (\sigma_i)_{1 \leq i \leq N}$  such that

$$\sigma_i = \begin{cases} 1 & \text{if } i \in V_{N,s_N}; \\ -1 & \text{if } i \in V_{N,r_N}; \\ \eta_i & \text{if } i \in V_{N,t_N}, \end{cases} \quad (1.6)$$

where  $\eta_i \in \{+1, -1\}$ . Thus all sites labelled by  $V_{N,s_N}$  have fixed spin value of  $+1$ , those on  $V_{N,r_N}$  are fixed to the spin value of  $-1$  and those labelled by  $V_{N,t_N}$  are free to choose any of the two spin values. Here  $+1$  corresponds to *opinion one* and  $-1$  to *opinion two*. Thus the part of the population indexed by  $V_{N,s_N}$  are those who have decided on opinion one and those with label set  $V_{N,r_N}$  are those who have chosen opinion two with those in  $V_{N,t_N}$  yet to decide. In this paper we study a conditional distribution  $\mu_{N,s,r}$  of the Curie-Weiss model  $\mu_N$  (1.1) conditional on spins in  $\Omega_{N,s,r}$ . Thus  $\mu_{N,s,r}$  is given by

$$\begin{aligned} \mu_{N,s,r}(\sigma) &= \mu_N(\sigma | \Omega_{N,s,r}) \\ &= \frac{\mu_N(\sigma)}{\mu_N(\Omega_{N,s,r})} \\ &= \frac{e^{-\beta H_N(\sigma)}}{\sum_{\eta \in \Omega_{N,s,r}} e^{-\beta H_N(\eta)}} \\ &= \frac{e^{-\beta H_N(\sigma)}}{\tilde{Z}_{N,s,r}} \end{aligned} \quad (1.7)$$

Thus  $\mu_{N,s,r}$  is a probability measure on  $\Omega_{N,s,r}$ . The above conditional probability is well defined as for all  $N \in \mathbb{N}$

$$\begin{aligned} \mu_N(\Omega_{N,s,r}) &= \frac{\sum_{\sigma \in \Omega_{N,s,r}} e^{-\beta H_N(\sigma)}}{\sum_{\tilde{\sigma} \in \Omega_N} e^{-\beta H_N(\tilde{\sigma})}} \\ &\geq 2^{|V_{N,t_N}| - N} e^{-\beta \sup_{\sigma, \tilde{\sigma} \in \Omega_N} |H_N(\sigma) - H_N(\tilde{\sigma})|} \\ &\geq 2^{|V_{N,t_N}| - N} e^{-\beta \left( \frac{N-1}{2} + 2N|h| \right)} \\ &> 0. \end{aligned} \quad (1.8)$$

We are ready to state the main results of the paper in the next subsection. Before we do this, let us define

$$m(\beta, s, r, h) = \lim_{N \rightarrow \infty} \frac{1}{N} \int_{\Omega_{N,s,r}} \left( \sum_{i=1}^N \sigma_i \right) d\mu_{N,s,r}(\sigma). \quad (1.9)$$

### 1.3 Main Results

In what follows, we always assume  $s, r \geq 0$  and  $s + r < 1$ .

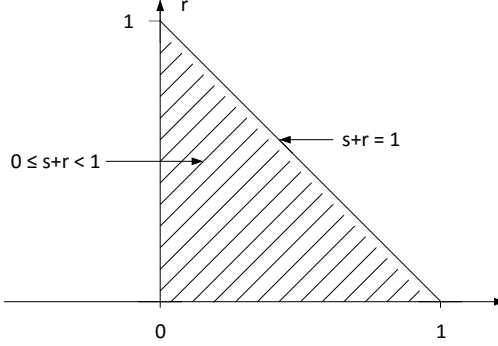


Figure 1: The shaded region in the  $sr$ -plane is the region of interest for this study.

**Theorem 1.2** For  $\beta > 0$ ,  $h \neq r - s$  and  $0 < \beta \leq (1 - s - r)^{-1}$ ,  $h = r - s$ , the specific magnetization  $m(\beta, s, r, h)$  is given by

$$m(\beta, s, r, h) = s - r + (1 - s - r)z(\beta, s, r, h), \quad (1.10)$$

where  $z(\beta, s, r, h)$  is the minimizer of the function

$$i_{\beta, s, r, h}(z) = -\frac{1}{2}\beta(1 - s - r)z^2 - \beta(s - r + h)z + \frac{1 - z}{2}\log(1 - z) + \frac{1 + z}{2}\log(1 + z), \quad (1.11)$$

for  $-1 \leq z \leq 1$ . In particular,

$$\begin{aligned} m(\beta, s, r, (r - s)^\pm) &= \lim_{h \rightarrow (r - s)^\pm} m(\beta, s, r, h) \\ &= \begin{cases} s - r, & \text{for } 0 < \beta \leq (1 - s - r)^{-1}, \\ s - r + (1 - s - r)z(\beta, s, r, (r - s)^\pm) \neq s - r, & \text{for } \beta > (1 - s - r)^{-1}, \end{cases} \end{aligned} \quad (1.12)$$

where

$$z(\beta, s, r, (r - s)^\pm) = \lim_{h \rightarrow (r - s)^\pm} z(\beta, s, r, h). \quad (1.13)$$

The following corollaries list what happens to the limit in (1.12) as we move through the shaded region of Figure 1.

**Corollary 1.3** Suppose  $s = r$  and  $0 \leq s < \frac{1}{2}$ . Then

$$\begin{aligned} m(\beta, s, s, 0^\pm) &= \lim_{h \rightarrow 0^\pm} m(\beta, s, s, h) \\ &= \begin{cases} 0, & \text{for } 0 < \beta \leq (1 - 2s)^{-1}, \\ (1 - 2s)z(\beta, s, s, 0^\pm) \neq 0, & \text{for } \beta > (1 - 2s)^{-1}. \end{cases} \end{aligned} \quad (1.14)$$

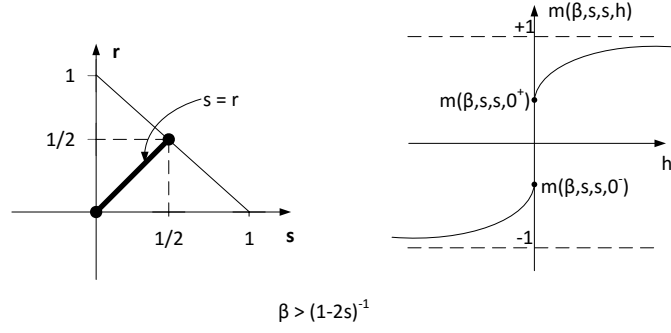


Figure 2: Discontinuity in the map  $h \mapsto m(\beta, s, r, h)$  at  $h = 0$  for  $\beta > (1 - 2s)^{-1}$  and  $s = r$ .

In particular, if  $\beta > (1 - 2s)^{-1}$  then there is a discontinuity in  $m(\beta, s, r, h)$ , as a function of  $h$ , at  $h = 0$  and this discontinuity is followed by change in state, i.e.  $m(\beta, s, s, 0^+) = -m(\beta, s, s, 0^-)$ .

**Corollary 1.4** Suppose  $\frac{1}{2} \leq s < 1$  or  $\frac{1}{2} \leq r < 1$  and  $\frac{1}{2} \leq s + r < 1$ . Then

$$m(\beta, s, r, (r-s)^\pm) = \lim_{h \rightarrow (r-s)^\pm} m(\beta, s, r, h) = \begin{cases} s - r, & \text{for } 0 < \beta \leq (1 - s - r)^{-1}, \\ s - r + (1 - s - r)z(\beta, s, r, (r-s)^\pm) \neq s - r, & \text{for } \beta > (1 - s - r)^{-1}. \end{cases} \quad (1.15)$$

Here there is a discontinuity in the map  $h \mapsto m(\beta, s, r, h)$  at  $h = r - s$  for  $\beta > (1 - s - r)^{-1}$ . This discontinuity is not followed by change in state, i.e.  $m(\beta, s, r, (r-s)^+)$  and  $m(\beta, s, r, (r-s)^-)$  are not equal but have the same sign as  $s - r$ .

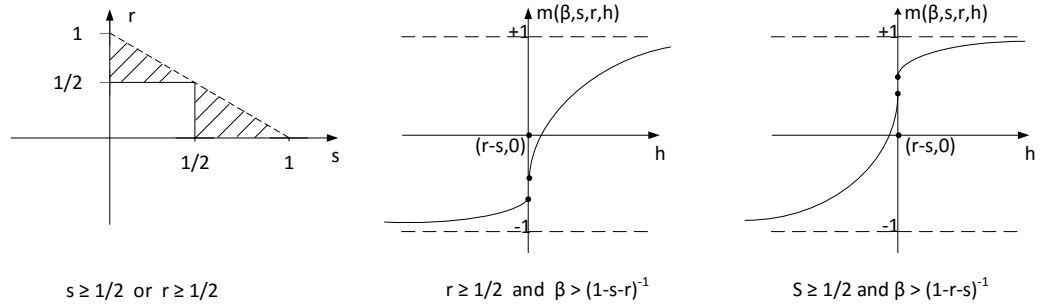


Figure 3: Discontinuity in the map  $h \mapsto m(\beta, s, r, h)$  at  $h = r - s$  for  $\beta > (1 - s - r)^{-1}$  and for choices of  $(s, r)$  from the shaded region in the  $sr$ -plane. The shaded region corresponding to  $\frac{1}{2} \leq r < 1$  has  $s - r < 0$  and that of  $\frac{1}{2} \leq s < 1$  has  $s - r > 0$ .

**Corollary 1.5** Suppose  $0 \leq s, r < \frac{1}{2}$  and  $s \neq r$ . Then

$$m(\beta, s, r, (r-s)^\pm) = \lim_{h \rightarrow (r-s)^\pm} m(\beta, s, r, h) = \begin{cases} s - r, & \text{for } 0 < \beta \leq (1 - s - r)^{-1}, \\ s - r + (1 - s - r)z(\beta, s, r, (r-s)^\pm) \neq s - r, & \text{for } \beta > (1 - s - r)^{-1}. \end{cases} \quad (1.16)$$

There is discontinuity in  $h \mapsto m(\beta, s, r, h)$  at  $h = r - s$  for  $\beta > (1 - s - r)^{-1}$ . Here we have the followed scenarios:

1. The discontinuity is not followed by change in state, i.e.  $m(\beta, s, r, (r - s)^+)$  and  $m(\beta, s, r, (r - s)^-)$  have the same sign as that of  $s - r$  for all

$$(1 - s - r)^{-1} < \beta < \frac{1}{r - s} \operatorname{arctanh} \left( \frac{r - s}{1 - s - r} \right). \quad (1.17)$$

2. For

$$\beta = \frac{1}{r - s} \operatorname{arctanh} \left( \frac{r - s}{1 - s - r} \right), \quad (1.18)$$

The discontinuity is followed by change from an ordered state (a plus or minus phase depending on the sign of  $s - r$ ) to a disordered state with zero specific magnetization.

3. Finally, for

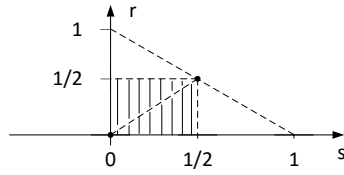
$$\beta > \frac{1}{r - s} \operatorname{arctanh} \left( \frac{r - s}{1 - s - r} \right), \quad (1.19)$$

the discontinuity is followed by change in state i.e.  $m(\beta, s, r, (r - s)^+) > 0$  and  $m(\beta, s, r, (r - s)^-) < 0$ .

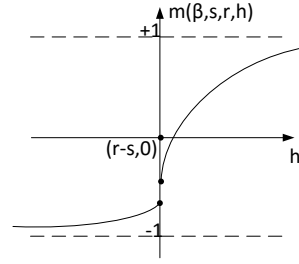
## 2 Discussion

1. As we will see in Section 3.1, the conditional Curie-Weiss model  $\mu_{N,s,r}$  is a Curie-Weiss model on the sites in  $V_{N,t_N}$  with  $N$ -dependent coupling strength  $1 - s_N - r_N$  and  $N$ -dependent external field  $h + s_N - r_N$ . Theorem 1.2 says that, up to a constant term  $s - r$  and a scaling factor  $1 - s - r$ , the large  $N$ -limit of the magnetization under  $\mu_{N,s,r}$  is determined by the magnetization of a Curie-Weiss model with coupling strength  $1 - s - r$  and external field  $h + s - r$ . The scaling factor  $1 - s - r$  is the effective size of the set of sites in  $V_{N,t_N}$ . Notice that if  $s = r = 0$ , then we get the original Curie-Weiss model with unit coupling strength and external field  $h$ .
2. If  $s = r \neq \frac{1}{2}$ , then the  $s - r$  part of the conditional magnetization vanishes. The conditional magnetization is then equal to the magnetization of the Curie-Weiss model with external field  $h$  and coupling strength scaled by the effective size  $1 - 2s$  of the conditional model. Corollary 1.3 says that in the regime where  $s = r \neq \frac{1}{2}$ , the singularity in the magnetization as  $h$  goes to zero and at  $\beta$ -values above  $(1 - 2s)^{-1}$  is similar to that of the original model only that in the conditional model the jump in the magnetization is suppressed by a factor of  $1 - 2s$ .
3. In the regime where either  $\frac{1}{2} \leq s < 1$  or  $\frac{1}{2} \leq r < 1$ , with  $s + r < 1$ , it is clear that the conditional model will always be negatively or positively magnetized depending on the sign of  $s - r$ . Though the conditional model is always magnetized along the sign of  $s - r$ , yet Corollary 1.4 indicates that the conditional magnetization discontinuously jumps as  $h$  tends to  $r - s$  at  $\beta$ -values greater than  $(1 - s - r)^{-1}$  (see Figure 3). This discontinuous jump does not lead to phase change.

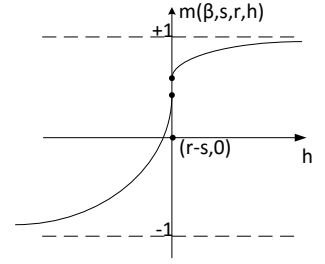
(a).



$$0 \leq s, r < 1/2$$



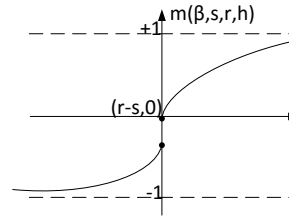
$$s-r < 0$$



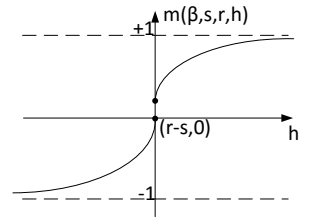
$$s-r > 0$$

$$(1-r-s)^{-1} < \beta < (1/(r-s)) \operatorname{arctanh}((r-s)/(1-r-s))$$

(b).



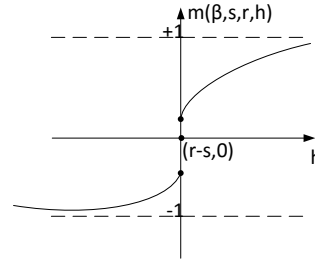
$$s-r < 0$$



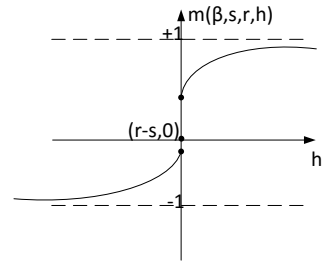
$$s-r > 0$$

$$\beta = (1/(r-s)) \operatorname{arctanh}((r-s)/(1-r-s))$$

(c).



$$s-r < 0$$



$$s-r > 0$$

$$\beta > (1/(r-s)) \operatorname{arctanh}((r-s)/(1-r-s))$$

Figure 4: *Discontinuity in the map  $h \mapsto m(\beta, s, r, h)$  at  $h = r - s$  for  $\beta > (1 - s - r)^{-1}$  and for choices of  $(s, r)$  from the shaded region in the  $sr$ -plane. The discontinuity in (a) does not lead to change in phase/behaviour. That in (b) results in transition from an ordered phase, with positive/negative magnetization, to disordered phase with zero magnetization. The situation in (c) leads to transition from positive magnetization to negative one or vice versa though  $m(\beta, s, r, (r - s)^-)$  and  $m(\beta, s, r, (r - s)^+)$  are not symmetric about zero.*



4. The most interesting region in the  $sr$ -plane is where  $0 \leq s, r < \frac{1}{2}$  and  $s \neq r$ . In this region, the conditional magnetization has three different forms of discontinuous jumps depending on the choice of  $\beta$ -value(s). Here there is a second transition point for  $\beta$  namely;

$$\beta^{**} = \frac{1}{r-s} \operatorname{arctanh} \left( \frac{r-s}{1-s-r} \right) \quad (2.1)$$

in addition to the earlier transition point

$$\beta^* = (1-s-r)^{-1}. \quad (2.2)$$

Note that  $\beta^* < \beta^{**}$ , for  $0 \leq s, r < \frac{1}{2}$  and  $s \neq r$ . Corollary 1.5 says the following:

- (a) For  $\beta \in (\beta^*, \beta^{**})$ , the discontinuity in the conditional magnetization is analogous to the case in Corollary 1.4. The conditional model is magnetized along the sign of  $s-r$ , with jump discontinuity at  $h = r-s$ . We have a discontinuous jump in an order parameter that is not accompanied by change in phase (see Figure 4(a)).
- (b) In the case  $\beta = \beta^{**}$ , there is also discontinuous jump in the magnetization. Here there is a nonzero magnetization as  $h$  approaches  $r-s$  from the values of  $h$  for which  $h+s-r$  has the same sign as  $s-r$ . On the other hand, there is no net magnetization as  $h$  goes to  $r-s$  from  $h$ -values for which  $s-r$  and  $h+s-r$  have different signs. Thus the discontinuity here leads to a change from an ordered phase, with a net magnetization, to a disordered phase, with no magnetization or vice versa (see Figure 4(b)).
- (c) Further, the discontinuous jump for the case  $\beta > \beta^{**}$  is similar to the case with  $s = r$  discussed in Corollary 1.3. This discontinuity is followed by change in phase, either from negatively magnetized phase to a positively magnetized phase or vice versa. Observe that in Corollary 1.3 the net magnetizations at the discontinuity point are symmetric about zero. Over here, the net magnetizations at the discontinuity point are not symmetric about zero, due to the presence of the term  $s-r$  (see Figure 4(c)).

## 3 Proofs

### 3.1 Proof of Theorem 1.2

*Proof.* Note from (1.6) that for any  $\sigma \in \Omega_{N,s,r}$

$$m_{N,s,r} = \frac{1}{N} \sum_{i=1}^N \sigma_i = s_N - r_N + \frac{1}{N} \sum_{i \in V_{N,t_N}} \sigma_i \quad (3.1)$$

Therefore

$$m_{N,s,r}^2 = s_N^2 + r_N^2 - 2s_N r_N + \left( \frac{1}{N} \sum_{i \in V_{N,t_N}} \sigma_i \right)^2 + 2s_N \frac{1}{N} \sum_{i \in V_{N,t_N}} \sigma_i - 2r_N \frac{1}{N} \sum_{i \in V_{N,t_N}} \sigma_i. \quad (3.2)$$

It follows from the last equality of (1.2), (3.1) and (3.2) that for any  $\sigma \in \Omega_{N,s,r}$ , the Hamiltonian  $H_N(\sigma)$  can be written as follows:

$$H_N(\sigma) = -N \left[ \frac{1}{2}(s_N^2 + r_N^2) - s_N r_N + h(s_N - r_N) + \frac{N}{2} \right] - H_{N,s,r}(\sigma) \quad (3.3)$$

where

$$\begin{aligned} H_{N,s,r}(\sigma) &= -N \left[ \frac{1}{2} \left( \frac{1}{N} \sum_{i \in V_{N,t_N}} \sigma_i \right)^2 + \frac{s_N - r_N + h}{N} \sum_{i \in V_{N,t_N}} \sigma_i \right] \\ &= \frac{|V_{N,t_N}|}{t_N} \left[ \frac{1}{2} \left( \frac{t_N}{|V_{N,t_N}|} \sum_{i \in V_{N,t_N}} \sigma_i \right)^2 + \frac{t_N[s_N - r_N + h]}{|V_{N,t_N}|} \sum_{i \in V_{N,t_N}} \sigma_i \right] \\ &= |V_{N,t_N}| \left[ \frac{t_N}{2} \left( \frac{1}{|V_{N,t_N}|} \sum_{i \in V_{N,t_N}} \sigma_i \right)^2 + \frac{s_N - r_N + h}{|V_{N,t_N}|} \sum_{i \in V_{N,t_N}} \sigma_i \right]. \end{aligned} \quad (3.4)$$

The second equality of (3.4) follows from our assumption that  $t_N = \frac{|V_{N,t_N}|}{N}$ . Comparing equality three of (3.4) with the second equality of (1.2) we observe that  $H_{N,s,r}$  has an  $N$ -dependent coupling strength of  $t_N$  as against 1 for  $H_N$ . Further,  $H_{N,s,r}$  has an  $N$ -dependent external field of  $h + s_N - r_N$  as against  $h$  for  $H_N$ . Therefore substituting this form of  $H_N$  in (3.3) into the expression for  $\mu_{N,s,r}$  in (1.7) leads to

$$\begin{aligned} \mu_{N,s,r}(\sigma) &= \frac{e^{-\beta H_{N,s,r}(\sigma)}}{\sum_{\eta \in \Omega_{N,s,r}} e^{-\beta H_{N,s,r}(\eta)}} \\ &= \frac{e^{-\beta H_{N,s,r}(\sigma)}}{Z_{N,s,r}}. \end{aligned} \quad (3.5)$$

Thus  $\mu_{N,s,r}$  is the Curie-Weiss model on the set of sites in  $V_{N,t_N}$  with an  $N$ -dependent external field  $h + s_N - r_N$  and  $N$ -dependent coupling strength  $t_N$ . In view of this, we have for  $\beta > 0$ ,  $h + s - r \neq 0$  and  $\beta \leq (1 - s - r)^{-1}$ ,  $h + s - r = 0$  that

$$\begin{aligned} m(\beta, s, r, h) &= \lim_{N \rightarrow \infty} \frac{1}{N} \int \left( \sum_{i \in V_N} \sigma_i \right) d\mu_{N,s,r}(\sigma) \\ &= \lim_{N \rightarrow \infty} \left[ s_N - r_N + \frac{1}{N} \int \left( \sum_{i \in V_{N,t_N}} \sigma_i \right) d\mu_{N,s,r}(\sigma) \right] \\ &= s - r + \lim_{N \rightarrow \infty} \frac{|V_{N,t_N}|}{N} \frac{1}{|V_{N,t_N}|} \int \left( \sum_{i \in V_{N,t_N}} \sigma_i \right) d\mu_{N,s,r}(\sigma) \\ &= s - r + (1 - s - r) \lim_{N \rightarrow \infty} \frac{1}{|V_{N,t_N}|} \int \left( \sum_{i \in V_{N,t_N}} \sigma_i \right) d\mu_{N,s,r}(\sigma) \\ &= s - r + (1 - s - r) z(\beta, s, r, h), \end{aligned} \quad (3.6)$$

where according to Theorem 1.1 and the last equality of (3.4),  $z(\beta, s, r, h)$  is the  $z$ -value in  $[-1, 1]$  that minimizes

$$i_{\beta, s, r, h}(z) = -\frac{1}{2}\beta(1-s-r)z^2 - (h+s-r)z + \frac{1+z}{2}\log(1+z) + \frac{1-z}{2}\log(1-z). \quad (3.7)$$

Next observe that for  $\beta \leq (1-s-r)^{-1}$ ,  $z(\beta, s, r, h)$  tends to zero as  $h$  goes to  $r-s$  and for  $\beta > (1-s-r)^{-1}$ ,  $z(\beta, s, r, h) \mapsto z(\beta, s, r, (r-s)^+) > 0$  as  $h \downarrow r-s$  and  $z(\beta, s, r, h) \mapsto z(\beta, s, r, (r-s)^-) < 0$  as  $h \uparrow r-s$ . ■

### 3.2 Proof of Corollary 1.3

*Proof.* Note that for  $s = r$ ,  $m(\beta, s, s, h) = (1-2s)z(\beta, s, s, h)$  and for  $\beta > (1-2s)^{-1}$  and  $h = 0$ ,  $i_{\beta, s, s, 0}(z)$  has two minimizers  $z(\beta, s, s, 0^+) > 0$  and  $z(\beta, s, s, 0^-) = -z(\beta, s, s, 0^+)$ . Further, as  $h \downarrow 0$ ,  $z(\beta, s, s, h) \rightarrow z(\beta, s, s, 0^+)$  and  $z(\beta, s, s, h) \rightarrow z(\beta, s, s, 0^-)$  as  $h \uparrow 0$ . ■

### 3.3 Proof of Corollary 1.4

*Proof.* As indicated in the proof of Corollary 1.3, the discontinuity in  $m(\beta, s, r, h)$  is created by the discontinuity in the  $z(\beta, s, r, h)$  part of  $m(\beta, s, r, h)$ . Now suppose we are in the regime where  $m(\beta, s, r, (r-s)^\pm) > 0$  no matter the choice of  $r$ , such that  $s+r < 1$ . Then the question is what range of  $s$ -values will permit this behaviour. First of all note that the largest possible jump that can occur in this case is when  $z(\beta, s, r, (r-s)^-) = -1$ . If at this value of  $z(\beta, s, r, (r-s)^-)$ ,  $m(\beta, s, r, (r-s)^-) > 0$ , then we have that  $s-r-(1-s-r) > 0$ , which leads to  $s > \frac{1}{2}$ . Thus if  $s > \frac{1}{2}$ , then the minimum value of  $z(\beta, s, r, (r-s)^-)$  is incapable of making  $m(\beta, s, r, (r-s)^-)$  nonpositive. Further, for  $s = \frac{1}{2}$ ,

$$m(\beta, s, r, (r-s)^-) = \left(\frac{1}{2} - r\right) [1 + z(\beta, s, r, (r-s)^-)] > 0,$$

for  $\beta \in (0, \infty)$  as  $r < \frac{1}{2}$  and  $z(\beta, s, r, (r-s)^-) > -1$ , for  $\beta \in (0, \infty)$ . Thus  $m(\beta, s, r, (r-s)^\pm) > 0$  for  $s \geq \frac{1}{2}$  and  $\beta > 0$ , though  $m(\beta, s, r, (r-s)^+) > m(\beta, s, r, (r-s)^-)$ . Similarly,  $m(\beta, s, r, (r-s)^\pm) < 0$ , for  $r \geq \frac{1}{2}$  and  $\beta > 0$ . ■

### 3.4 Proof of Corollary 1.5

*Proof.* Observe that the  $|m(\beta, s, r, (r-s)^\pm)|$  is an increasing function of  $\beta$ , with codomain  $[-1, 1]$ . Suppose  $0 \leq r < s < \frac{1}{2}$ . The question now is at what value of  $\beta$  is  $m(\beta, s, r, (r-s)^-) = 0$ , thus

$$z(\beta, s, r, (r-s)^-) = \frac{r-s}{1-s-r}. \quad (3.8)$$

Further,  $z(\beta, s, r, (r-s)^-)$  is one of the minimizers of the function  $z \mapsto i_{\beta, s, r, r-s}(z)$  given by

$$i_{\beta, s, r, r-s}(z) = -\frac{\beta}{2}(1-s-r)z^2 + \frac{1+z}{2}\log(1+z) + \frac{1-z}{2}\log(1-z), \quad \text{for } z \in [-1, 1].$$

These minimizers satisfy the self-consistency equation

$$z = \tanh(\beta[1-s-r]z). \quad (3.9)$$

Therefore, it follows from (3.8) and (3.9) that the minimizer  $z$  for which  $m(\beta, s, r, (r-s)^-) = 0$  must satisfy

$$\frac{r-s}{1-s-r} = \tanh(\beta(r-s)).$$

This gives rise to the following expression for  $\beta$ :

$$\beta = \frac{1}{r-s} \operatorname{arctanh}\left(\frac{r-s}{1-s-r}\right). \quad (3.10)$$

Since  $\beta \mapsto |m(\beta, s, r, (r-s)^-)|$  is increasing, we have the following:

1. For  $\frac{1}{1-s-r} < \beta < \frac{1}{r-s} \operatorname{arctanh}\left(\frac{r-s}{1-s-r}\right)$ , there is a discontinuity in the map  $h \mapsto m(\beta, s, r, h)$ , at  $h = r-s$ , though both  $m(\beta, s, r, (r-s)^-)$  and  $m(\beta, s, r, (r-s)^+)$  will be positive.
2. At  $\beta = \frac{1}{r-s} \operatorname{arctanh}\left(\frac{r-s}{1-s-r}\right)$ , we experience discontinuity at  $h = r-s$  but this time round  $m(\beta, s, r, (r-s)^-) = 0$  and  $m(\beta, s, r, (r-s)^+) > 0$ .
3. Similar result holds for  $\beta > \frac{1}{r-s} \operatorname{arctanh}\left(\frac{r-s}{1-s-r}\right)$ , with  $m(\beta, s, r, (r-s)^-) < 0$  and  $m(\beta, s, r, (r-s)^+) > 0$ .

The proof for the case  $0 \leq s < r < \frac{1}{2}$  follows from arguments similar to the case considered above. ■

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